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SOLUTION OF THE REYNOLDS-AVERAGED EQUATIONS FOR TURBULENT FLOW VIA INTEGRAL TRANSFORM AND ALGEBRAIC TURBULENCE MODEL

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Abstract. *The Generalized Integral Transform Technique (GITT) is utilized in the hybrid numerical-analytical solution of the Reynolds-averaged boundary layer equations, for developing turbulent flow inside a parallel-plates channel. An algebraic turbulence model, a modified version of the local Van Driest effective viscosity model, is employed in modelling the turbulent diffusivity. The streamfunction-only formulation is utilized, being solved by an eigenfunctions expansion obtained from the homogeneous bi-harmonic problem, associated to the original problem. Therefore, following the ideas in previous contributions to GITT, numerical results for different Reynolds number are obtained, both for illustrating the convergence characteristics of the integral transform approach, and for critical comparisons with previously reported results through different models and numerical schemes.*

Keywords: *Turbulent channel flow, Algebraic model, Integral transform*

1. INTRODUCTION

The analysis of a developing turbulent flow inside parallel-plates channels finds numerous applications in the engineering practice, starting from the primary need of estimating pressure drops and, consequently, designing pumping power requirements. Such geometry is fundamental in the design of plate heat exchanger, in some nuclear reactor configurations, in air conditioning applications, in the cooling of microelectronic circuit boards, and as a limiting case of annular passage.

Despite the quite recent and progressive developments on the direct simulation of turbulence, the concept of Reynolds averaging, and associated turbulence modelling for closure, remains a more practical tool in engineering simulations, Wilcox (1984). The wellknown Van Driest model offers the basis for most of the algebraic models utilized in the literature, including the quite successful variations proposed by Herring and Mellor (1968) and Cebeci and Smith (1974). These applications were initially direct to be employed in conjunction with boundary layer formulations of governing flow equations. Later on, based on the modifications introduced by Hirst and Coles (1968), algebraic models were utilized in association with Reynolds-averaged Navier-Stokes equations, within an elliptical formulation, Richman and Azad (1973) and Taylor *et al.* (1977).

The so-called Generalized Integral Transform Technique is a spectral-type approach, based on eigenfunction expansions that incorporate some ingredients of a classical analytical approach with the aid of symbolic algebraic manipulation packages, Cotta (1993) and Cotta and Mikhailov (1997). Due to its inherently hybrid nature, this technique presents some interesting features such as the automatic and straightforward global error control procedure, which make it particularly suitable for benchmarking purposes; as well as only a mild increase in overall computational effort with an increase number of independent variables. Following the successful implementations of this approach in the solution of the equations for laminar flow situations, the developing turbulent flow between two parallel-plates was recently studied through the GITT, (Pimentel and Cotta, 1998), employing a boundary layer formulation and the algebraic turbulence model proposed in Cebeci and Smith (1974).

As a natural sequence to the contribution to the study of turbulent channel flow, the present work progress towards the integral transform solution of the Reynolds-averaged boundary layer equations, for a parallel-plates configuration, as in Pimentel and Cotta (1998). However, another algebraic turbulence model is utilized, that one employed by Richman and Azad (1973) and Taylor *at al.* (1977), following both the finite differences and finite elements methods. The convergence characteristics of the approach are illustrated for different values of Reynolds number, and critical comparisons against previously reported simulations are performed, in an attempt to elucidate merits and deficiencies of the modeling adopted.

2. PROBLEM FORMULATION

We consider the incompressible two-dimensional turbulent flow of a Newtonian fluid with constant physical properties developing between parallel-plates, as depicted in "Fig. 1".

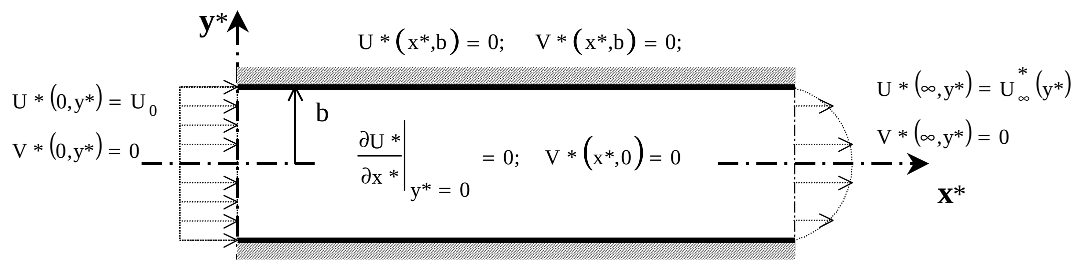

Figure 1. Geometry, coordinate system, inlet and boundary conditions

The fluid enters the channel under a uniform and parallel flow, and the laminar-turbulent transition is assumed to occur right at the duct entrance. The average flow is considered to be in steady state and the concept of turbulent viscosity is adopted.

After an application of the Boussinesq hypothesis, and employing the following dimensionless quantities:

$$x = \frac{x^*}{b}; \quad y = \frac{y^*}{b}; \quad U = \frac{U^*}{\overline{U}}; \quad V = \frac{V^*}{\overline{U}}; \quad Re = \frac{\overline{U}b}{\nu}; \quad \nu_t = \frac{\nu_t^*}{\nu}; \quad P = \frac{1}{\overline{U}^2}\left[\frac{P^*}{\rho} + \frac{2}{3}K^*\right]$$

the continuity and averaged boundary layer Reynolds equations are written, in the primitive variable formulation and in dimensionless form, as:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad , \qquad 0 < y < 1 \quad , \qquad x > 0 \tag{1}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re}\frac{\partial}{\partial y}\left[(1+\nu_t)\frac{\partial U}{\partial y}\right] \tag{2}$$

$$0 = -\frac{\partial P}{\partial y} \tag{3}$$

subjected to the inlet and boundary conditions, respectively:

$$x = 0, \quad 0 \le y \le 1$$

$$U(0, y) = 1 \tag{4}$$

$$V(0, y) = 0 \tag{5}$$

$$x > 0, \quad y = 0 \qquad\qquad x > 0, \quad y = 1$$

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = 0 \qquad U(x,1) = 0 \qquad V(x,0) = 0 \qquad V(x,1) = 0 \tag{6-9}$$

It has been demonstrated in previous contributions on the integral transform method that the streamfunction-only formulation for two-dimensional flow, offers some advantages over the more usual primitive variables version, including the automatic satisfaction of the continuity equation and elimination of the pressure gradients. Therefore, "Eqs. (1-9)" are now rewritten in the streamfunction-only formulation, starting from its definition:

$$U(x, y) = \frac{\partial \Psi(x, y)}{\partial y} \quad ; \quad V(x, y) = -\frac{\partial \Psi(x, y)}{\partial x} \tag{10-11}$$

which yields, after the appropriate manipulation with "Eqs. (2-3)", the governing equation for the dimensionless streamfunction:

$$\frac{\partial \Psi}{\partial y}\frac{\partial^3 \Psi}{\partial x \partial y^2} - \frac{\partial \Psi}{\partial x}\frac{\partial^3 \Psi}{\partial y^3} = \frac{1}{Re}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left((1+\nu_t)\frac{\partial^2 \Psi}{\partial y^2}\right)\right] \tag{12}$$

subjected to the inlet and boundary conditions, respectively:

$$x = 0, \quad 0 \le y \le 1$$

$$\Psi(0, y) = y \tag{13}$$

$$\left.\frac{\partial \Psi}{\partial x}\right|_{x=0} = 0 \tag{14}$$

$x > 0,\quad y = 0$ $\qquad\qquad$ $x > 0,\quad y = 1$

$$\Psi(x,0) = 0 \qquad\qquad \Psi(x,1) = 1$$

$$\left.\frac{\partial^2\Psi}{\partial y^2}\right|_{y=0} = 0 \qquad\qquad \left.\frac{\partial\Psi}{\partial y}\right|_{y=1} = 0 \tag{15-18}$$

2.1 Turbulence Model

According to Richman e Azad (1973), among the various models available for the effective viscosity in turbulent flows the Van Driest model offers the best compromise between simplicity and generality. The most important advantages are in avoiding any need for evaluating velocity gradients and artificially defined boundary layer thickness, within the turbulence model itself. The model proposed by Richman and Azad (1973), and utilized by Taylor *et al.* (1977) in conjunction with the full Navier-Stokes formulation for circular tubes, is written in dimensionless form as:

$$\nu_t = \frac{1}{2}\left\{1 + \sqrt{1 + 4K^2\,Y^{+2}\left(1 - e^{-Y^+/A}\right)^2}\right\} - 1 \quad ; \quad \begin{array}{l} 0 \le x \le L \\ 0 \le (1-y) \le 0.158 \end{array} \tag{19}$$

$$\nu_t = \left.\nu_t\right|_{1-y=0,158} \quad ; \quad \begin{array}{l} 0 \le x \le L \\ 0.158 \le (1-y) \le 1 \end{array} \tag{20}$$

where:

L	dimensionless hydrodynamic development length
$K = 0{,}41$	von Karman's constant
$A = 26$	damping constant of the viscous sublayer
$Y^+ = \dfrac{(b - y^*)\cdot u_\tau^*}{\nu}$	turbulent Reynolds number
$u_\tau^* = \sqrt{\dfrac{\tau_w^*}{\rho}}$	local friction velocity, dimensional
$\tau_w^* = -\mu \left.\dfrac{\partial U^*}{\partial y^*}\right\|_{y^*=b}$	wall shear stress, dimensional
$u_\tau = \sqrt{-\dfrac{1}{Re}\left.\dfrac{\partial^2\Psi}{\partial y^2}\right\|_{y=1}}$	local friction velocity, dimensionless
$Y^+ = Re\cdot(1-y)\cdot u_\tau$	turbulent Reynolds number, in terms of the dimensionless parameters

It can be noticed that this model is indeed quite simple, representing a truncation of the original Van Driest model at $(1-y) = 0{,}158$, yielding an uniform turbulent viscosity in the region $0{,}158 \le (1-y) \le 1$, and allowing for the dependence on the turbulent Reynolds number based on the local friction velocity, so as to account for the nonlinear pressure gradient.

3. SOLUTION METHODOLOGY

Following the ideas in the GITT, the boundary conditions in the direction to be integral transformed are, first of all, made homogeneous. For this purpose, a filtering solution is proposed, in this case the fully developed flow profile, $\Psi_\infty(y)$, which also analytically recovers the original solution for large x, in the form:

$$\Psi(x,y) = \Phi(x,y) + \Psi_\infty(y) \tag{21}$$

Then, "Eq. (12)" is rewritten for the filtered potential, $\Phi(x,y)$, as:

$$\left(\frac{\partial\Phi}{\partial y} + \frac{d\Psi_\infty}{dy}\right)\frac{\partial^3\Phi}{\partial x\partial y^2} - \frac{\partial\Phi}{\partial x}\left(\frac{\partial^3\Phi}{\partial y^3} + \frac{d^3\Psi_\infty}{dy^3}\right) = \frac{1}{Re}\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left[(1+\nu_t)\left(\frac{\partial^2\Phi}{\partial y^2} + \frac{d^2\Psi_\infty}{dy^2}\right)\right]\right\} \tag{22}$$

while the initial and boundary conditions, "Eqs. (13-18)", are similarly filtered, to provide:

$x = 0, \quad 0 \le y \le 1$

$$\Phi(0,y) = y - \Psi_\infty(y) \tag{23}$$

$$\left.\frac{\partial\Phi}{\partial x}\right|_{x=0} = 0 \tag{24}$$

$x > 0, \quad y = 0$ $\qquad$ $x > 0, \quad y = 1$

$$\Phi(x,0) = 0 \qquad \Phi(x,1) = 0$$

$$\left.\frac{\partial^2\Phi}{\partial y^2}\right|_{y=0} = 0 \qquad \left.\frac{\partial\Phi}{\partial y}\right|_{y=1} = 0 \tag{25-28}$$

The filtered system, "Eqs. (22-28)", is now in the appropriate form for integral transformation in the transversal direction, y, after extraction of the fully developed solution. The eigenvalue problem, employed as auxiliary problem for integral transformation, is obtained of a homogeneous version of the original problem, "Eq. (12)". It is presented in the work of Perez-Guerrero and Cotta (1995), and given by:

$$\frac{d^4\tilde{Y}_i(y)}{dy^4} = \mu_i^4\tilde{Y}_i(y) \qquad ; \qquad 0 < y < 1 \tag{29}$$

$$\tilde{Y}_i(0) = 0 \qquad \tilde{Y}_i(1) = 0$$

$$\left.\frac{d^2\tilde{Y}_i}{dy^2}\right|_{y=0} = 0 \qquad \left.\frac{d\tilde{Y}_i}{dy}\right|_{y=1} = 0 \tag{30-33}$$

This problem has the following analytical solution:

$$\tilde{Y}_i(y) = \frac{Sen(\mu_i\ y)}{Sen(\mu_i)} - \frac{Senh(\mu_i\ y)}{Senh(\mu_i)} \qquad ; \qquad i = 1,2,3... \tag{34}$$

while the associated eigenvalues, μ_i's, are evaluated from the transcendental equation:

$$\mathrm{Tanh}(\mu_i) = \mathrm{Tan}(\mu_i) \tag{35}$$

and the normalization integral gives $N_i = 1$.

The eigenvalue problem above allows the definition of the integral transform pair:

$$\Phi(x,y) = \sum_{i=1}^{\infty} \tilde{Y}_i(y)\ \overline{\Phi}_i(x) \qquad \text{Inverse} \tag{36}$$

$$\overline{\Phi}_i(x) = \int_0^1 \tilde{Y}_i(y)\ \Phi(x,y)\ dy \qquad \text{Transform} \tag{37}$$

where the proposed eigenfunction expansion appears in the inversion formula, "Eq. (36)", in terms of the transformed potentials, $\overline{\Phi}_i(x)$.

"Equation (22)" is now operated on with $\int_0^1 \tilde{Y}_i(y)\,dy$ to yield, after employing the inversion formula, "Eq. (36)", on terms with differentiation in x:

$$\begin{aligned}&\sum_{k=1}^{\infty}\left[\int_0^1 \tilde{Y}_i \tilde{Y}_k^{''}\left(\frac{\partial\Phi}{\partial y}+\frac{d\Psi_\infty}{dy}\right)dy - \int_0^1 \tilde{Y}_i \tilde{Y}_k\left(\frac{\partial^3\Phi}{\partial y^3}+\frac{d^3\Psi_\infty}{dy^3}\right)dy\right]\frac{d\overline{\Phi}_k}{dx} \\ &= \frac{1}{\mathrm{Re}}\int_0^1 \tilde{Y}_i^{''}(1+\nu_t)\frac{\partial^2\Phi}{\partial y^2}dy + \frac{1}{\mathrm{Re}}\int_0^1 \tilde{Y}_i^{''}(1+\nu_t)\frac{d^2\Psi_\infty}{dy^2}dy\end{aligned} \tag{38}$$

In compact form, the above equation can be rewritten as:

$$\sum_{k=1}^{\infty} A_{ik}\frac{d\overline{\Phi}_k}{dx} = B_i \qquad i = 1, 2, 3, \ldots \tag{39}$$

where the two coefficients are defined as:

$$A_{ik} = \sum_{j=1}^{\infty}\left[A_{ijk} - B_{ijk}\right]\overline{\Phi}_j + \left[C_{ik\infty} - D_{ik\infty}\right] \tag{40}$$

$$A_{ijk} = \int_0^1 \tilde{Y}_i \tilde{Y}_j^{'} \tilde{Y}_k^{''}\,dy \qquad B_{ijk} = \int_0^1 \tilde{Y}_i \tilde{Y}_j^{'''} \tilde{Y}_k\,dy \tag{41-42}$$

$$C_{ik\infty} = \int_0^1 \tilde{Y}_i \tilde{Y}_k^{''} \Psi_\infty^{'}\,dy \qquad D_{ik\infty} = \int_0^1 \tilde{Y}_i \tilde{Y}_k \Psi_\infty^{'''}\,dy \tag{43-44}$$

$$B_i = \frac{1}{\mathrm{Re}}\left[\mu_i^4\,\overline{\Phi}_i + B_{i\infty} + B_{i\nu\infty}\right] \tag{45}$$

$$B_{i\infty} = \int_0^1 \tilde{Y}_i^{''}\Psi_\infty^{''}\,dy \qquad B_{i\nu\infty} = \int_0^1 \tilde{Y}_i^{''}\nu_t\left(\frac{\partial^2\Phi}{\partial y^2}+\Psi_\infty^{''}\right)dy \tag{46-57}$$

The inlet condition is similarly integral transformed, to yield:

$x = 0, \quad 0 \le y \le 1$

$$\overline{\Phi}_i(0) = \int_0^1 \tilde{Y}_i \left[y - \Psi_\infty(y) \right] dy \qquad i = 1, 2, 3, \ldots \tag{48}$$

$$\left.\frac{d\overline{\Phi}_i}{dx}\right|_{x=0} = 0 \qquad i = 1, 2, 3, \ldots \tag{49}$$

The integral transformation process shall then eliminate the transversal coordinate, y, and offers an ordinary differential system for the transformed potentials. The infinity system, "Eqs. (39-49)", is now truncated to a sufficiently large finite order, N, in order to achieve numerical results to within an user prescribed accuracy target, through well-established subroutines for initial value problems, such as routine DIVPAG (IMSL, 1987). Once these quantities have been numerically evaluated for any axial position, x, the streamfunction, the velocity components and the friction factor are recovered analytically, by recalling their definitions and the inversion formula.

4. RESULTS AND DISCUSSION

The computational procedure was implemented in a Fortran code and executed on microcomputer Pentium II of 300 MHz. A relative error target of 10^{-5} (five significant digits precision) was prescribed in the call of subroutine DIVPAG, and the fully converged results are expected to be correct to within ± 1 in the last digit provided.

Numerical results for the streamfunction and longitudinal velocity component are now reported, for different values of the Reynolds number, aimed at illustrating the convergence characteristics of the proposed eigenfunction expansion, and allowing for critical comparisons with previously reported numerical and experimental findings.

"Table 1" illustrate the convergence behavior of the streamfunction profiles at selected axial positions (x^*/D_h = 10 and 25) in units of hydraulic diameter, for Re = 3,5 x 10^4. The excellent convergence rates are clearly observable from this set of results, with full convergence to three or four digits at quite low truncation orders.

Table 1. Convergence behavior of the streamfunction profiles, $\Psi(x, y)$, at positions x^*/D_h = 10 and 25 (Re = 3,5 x 10^4)

	x^*/D_h = 10				x^*/D_h = 25			
y | N	5	40	80	100	5	40	80	100
0,1	0,1106	0,1101	0,1101	0,1101	0,1129	0,1128	0,1127	0,1128
0,3	0,3301	0,3286	0,3286	0,3286	0,3359	0,3355	0,3355	0,3355
0,5	0,5438	0,5417	0,5416	0,5416	0,5503	0,5499	0,5498	0,5498
0,7	0,7460	0,7439	0,7437	0,7437	0,7504	0,7500	0,7499	0,7500
0,9	0,9290	0,9279	0,9277	0,9277	0,9300	0,9298	0,9298	0,9298
0,95	0,9695	0,9689	0,9687	0,9687	0,9698	0,9697	0,9697	0,9697
0,99	0,9968	0,9968	0,9967	0,9967	0,9968	0,9968	0,9968	0,9968

"Table 2" complement such observation, now for the longitudinal velocity component. As expected, as the fully developed region is approached, the convergence progressively improves, due to the filtering effect introduced by the fully developed analytical solution. It may be observed that convergence in the third digit is attained for a truncation order of N = 40 at position near the channel inlet, with a clear slower convergence rate near the wall.

Table 2. Convergence behavior of the longitudinal velocity component, $U(x, y)$, at positions $x^*/D_h = 10$ and 25 ($Re = 3{,}5 \times 10^4$)

	$x^*/D_h = 10$				$x^*/D_h = 25$			
y | N	5	40	80	100	5	40	80	100
0,0	1,106	1,101	1,101	1,102	1,130	1,129	1,129	1,129
0,1	1,105	1,099	1,099	1,100	1,127	1,125	1,125	1,125
0,3	1,087	1,083	1,083	1,083	1,098	1,098	1,097	1,097
0,5	1,046	1,044	1,043	1,043	1,041	1,041	1,041	1,041
0,7	0,9694	0,9725	0,9722	0,9720	0,9545	0,9553	0,9554	0,9553
0,9	0,8446	0,8526	0,8532	0,8532	0,8289	0,8302	0,8305	0,8304
0,95	0,7638	0,7765	0,7779	0,7782	0,7539	0,7555	0,7558	0,7558
0,99	0,5165	0,5274	0,5338	0,5353	0,5142	0,5153	0,5161	0,5162

"Table 3" presents the convergence for the longitudinal velocity component at the channel centerline, for different positions x^*/D_h and different Reynolds number, $Re = 3{,}5x10^4$ and $Re = 5{,}0x10^4$. These results reconfirm the excellent convergence characteristics, with full convergence to three digits in all cases, for N < 40. It is shown that the system truncation order N = 5 can represent satisfactory the fully converged results. In addition, the increase in Reynolds number does not appear to affect convergence rates significantly, in light of the variable turbulent viscosity behavior, which to a certain extent counterbalances the increased importance of the inertial terms.

Table 3 - Convergence behavior of the centerline longitudinal velocity, $U(x,0)$, at different positions along the channel and different Reynolds number

	$Re = 3{,}5 \times 10^4$				$Re = 5{,}0 \times 10^4$			
x^*/D_h | N	5	40	80	100	5	40	80	100
5	1,069	1,066	1,066	1,067	1,065	1,061	1,061	1,061
10	1,106	1,101	1,101	1,102	1,101	1,095	1,095	1.095
15	1,122	1,118	1,118	1,118	1,116	1,112	1,112	1,112
20	1,128	1,125	1,125	1,125	1,122	1,120	1,119	1,119
25	1,130	1,129	1,129	1,129	1,125	1,123	1,123	1,123
30	1,131	1,130	1,130	1,130	1,126	1,125	1,125	1,125
40	1,132	1,131	1,131	1,132	1,126	1,126	1,126	1,126
50	1,132	1,132	1,132	1,132	1,127	1,127	1,127	1,127

"Figure 1.a" brings a comparison of the present integral transform results and the experimental results of Byrne *et al.* (1969-70), for the longitudinal velocity component profiles at different axial positions and $Re = 3{,}5 \times 10^4$. Also, some samples results from previous numerical works are shown, such as the integral transform implementation in Pimentel and Cotta (1998) and the finite volume solution with k-ε model proposed in Zaparoli (1989). "Figure 1.b" shows a comparison among the present integral transform results from Pimentel and Cotta (1998) and the experimental results of Dean (1972), for the longitudinal velocity component profiles. The value of $Re = 5{,}0 \times 10^4$ was employed in these comparisons, and the velocity profiles are shown for different axial distances.

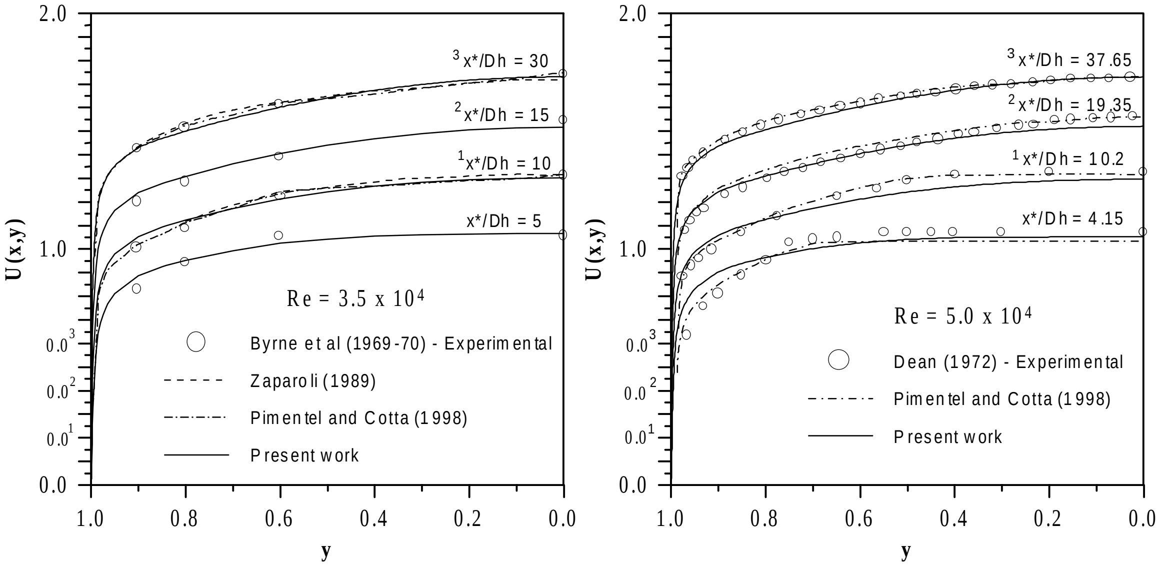

Figure 1.a, b - Development of longitudinal velocity component along channel and comparison against experimental and numerical results.

The overall agreement is quite reasonable, with some noticeable deviations of the present results from the experimental findings, at the wall region, for lower values of x^*/D_h, but improving for larger values of the longitudinal position. Also, within an intermediate range, the centerline velocity appears less adherent to the experiments, which might be an indication of a turbulence model limitation, since the more refined modeling in Pimentel and Cotta (1998) and Zaparoli (1989) presents a more consistent behavior.

To better demonstrate the limitation on the turbulence model employed here, "Figs 2.a, b" show the evolution of the centerline longitudinal velocity component along the channel dimensionless coordinate, x^*/D_h, respectively, for Re = 3,5 x 10^4 and Re = 5,0 x 10^4. Besides the experimental results of Byrne *et al.* (1969-70) and Dean (1972), alternative simulations with more involved turbulence models are presented, extracted from Pimentel and Cotta (1998), Bradshaw *et al.* (1973) and Emery and Gessner (1976).

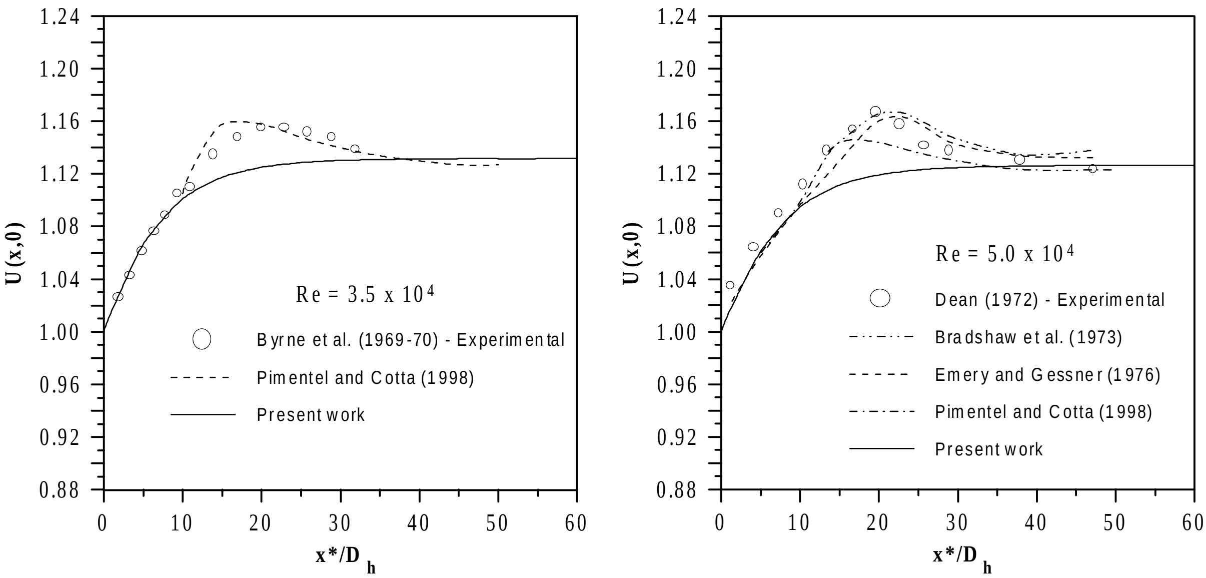

Figure 2.a, b - Development of the longitudinal velocity component at channel centerline, and comparison against experimental and numerical results.

5. CONCLUSIONS

Clearly, the very simple turbulence model utilized can adequately reproduce the centerline velocity at regions close to the channel inlet and as the fully developed region is approached. However, it leads to an underestimation in the region of the boundary layers interaction, since it does not introduce any empirical correction within this region, such as proposed by Cebeci and Smith (1974) and employed in Pimentel and Cotta (1998). For this reason, the non-asymptotic behavior of the centerline longitudinal velocity, as expected from the experimental observations, is not appropriately reproduced through the present algebraic model. Nevertheless, the overall behavior of the turbulent flow is reasonably well simulated and most important, in light of the global error control capability of the present approach, a set of reference results is offered in this work based on a rather simple turbulence modelling that can be easily implemented in any other numerical scheme, for validation purposes.

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